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THE SPECTRUM OF SUPPORT SIZES FOR THREEFOLD TRIPLE SYSTEMS

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The spectrum of possible numbers of distinct blocks in a threefold triple system of order v is determined. Let $m_v = \lfloor v(v-1)/6 \rfloor$. A threefold triple system with $v \equiv 1, 3 \pmod 6$ elements can have any number of distinct blocks from, and only from, $\{m_v, m_v+4, m_v+6, m_v+7, \cdots, 3m_v\}$ provided $v \neq 3, 7, 9$. A threefold triple system with $v \equiv 5 \pmod 6$ elements can have any number of distinct blocks from, and only from, $\{m_v+7, m_v+10, m_v+11, \cdots, 3m_v+1\}$.

1. The background

A *triple system of order* v and *index* λ, denoted $\mathrm{TS}(v, \lambda)$, is a collection of 3-element subsets $\mathscr{B}$ of a v-set V, so that every 2-subset of V appears in precisely λ subsets of $\mathscr{B}$. $\mathscr{B}$ is a multiset, and the set $\mathscr{B}^*$ of distinct blocks is the *support* of the triple system. While the number of blocks $|\mathscr{B}|$ always equals $\frac{\lambda}{3}\binom{v}{2}$, the *support size* $|\mathscr{B}^*|$, is dependent on the actual triple system. We let

$$\mathrm{SS}(v, \lambda) = \{k : \exists \mathrm{TS}(v, \lambda)(V, \mathscr{B}) \text{ with } |\mathscr{B}^*| = k\}.$$

Elementary counting shows that the smallest value in $\mathrm{SS}(v, \lambda)$ must be at least

$$m_v = \left\lfloor \frac{v(v-1)}{6} \right\rfloor.$$

Recently, there has been substantial interest in the spectrum of possible support sizes. Foody and Hedayat [4] introduced this problem, and since that time computational results for triple systems with $v \leq 10$ have appeared in [7–11]. Results for $v = 10$ and $\lambda = 2$ can be obtained more directly from the complete list of nonisomorphic $\mathrm{TS}(10, 2)$ designs with repeated blocks [5].

The only general result on support sizes is for $\mathrm{TS}(v, 2)$ designs. In a $\mathrm{TS}(v, 2)$ design, the support size plus the number of repeated blocks is just the total number of blocks. Lindner and Rosa [14] prove a theorem which establishes the following:

Theorem A [14]. *For* $v \equiv 1, 3 \pmod 6$, $\mathrm{SS}(v, 2) = \{m_v, m_v + 4, m_v + 6, m_v + 7, \cdots, 2m_v\}$ *provided* $v \geq 13$.

Rosa and Hoffman [19] determined the spectrum of repeated blocks in TS(v, 2) with $v \equiv 0, 4 \pmod 6$, hence completing the determination of SS(v, 2). Rosa [18] subsequently determined the spectrum of all support sizes of indecomposable TS(v, 2) designs.

The results for $\lambda = 2$ can be viewed either as determining the number of repeated blocks, or as determining the number of distinct blocks. Hence one extension of interest is to determine the number of three-times repeated blocks in TS(v, 3) designs. Milici and Quattrochi [16] address this question and prove the following:

Theorem B [16]. *In a* TS(v, 3), *the number of three-times repeated blocks is any number from* $\{0, 1, \ldots, m_v - 7, m_v - 6, m_v - 4, m_v\}$ *for* $v \equiv 1, 3 \pmod 6$ *and* $v \neq 9$, *and is any number from* $\{0, \ldots, m_v - 6, m_v - 3\}$ *provided* $v \equiv 5 \pmod 6$ *and* $v \geq 17$.

Milici and Quattrochi [17] also consider the related question of determining the number of blocks which three Steiner triple systems can have in common.

We consider the "distinct blocks" extension instead. In this paper, we completely determine SS(v, 3) for all values of v. Our main theorem is the following:

Main Theorem. *For* $v \equiv 1, 3 \pmod 6$, $\mathrm{SS}(v, 3) = P_v = \{m_v, m_v + 4, m_v + 6, \cdots, 3m_v\}$ *provided* $v \neq 3, 7, 9$; $\mathrm{SS}(3, 3) = \{1\}$, $\mathrm{SS}(7, 3) = \{7, 11, 13\text{–}15, 17\text{–}21\}$ *and* $\mathrm{SS}(9, 3) = \{12, 18, 20\text{–}36\}$. *For* $v \equiv 5 \pmod 6$, $\mathrm{SS}(v, 3) = Q_v = \{m_v + 7, m_v + 10, \cdots, 3m_v + 1\}$.

The remainder of the paper is a proof of the main theorem. In Section 2, we establish necessity of the conditions. In Section 3, we provide recursive constructions for creating threefold triple systems with desired support sizes. In Section 4, we examine the required small cases.

2. Necessity of the conditions

In this section, we consider necessary conditions for $b^* \in \mathrm{SS}(v, 3)$. The blocks of the support $\mathcal{B}^*$ of a TS(v, 3) may appear once, twice, or three times, in the collection $\mathcal{B}$ of blocks; let b_1, b_2, and b_3 denote the number of blocks of these types, respectively. Then we have

$$b_1 + 2b_2 + 3b_3 = \binom{v}{2} \tag{1}$$

from counting unordered pairs. We also have

$$b_1 + b_2 + b_3 = b^* \tag{2}$$

for $b^* \in \mathrm{SS}(v, 3)$. It is also an easy exercise to see that we have

$$b_1 \geqslant b_2 \tag{3}$$

by considering the pairs remaining from the two-times repeated blocks. Theorem B tells us the possible values of b_3, and hence we need to determine the possible values of $b_1 + b_2$ for a given value of b_3.

Let us suppose that $b_3 = m_v - s$. Of the $\binom{v}{2}$ 2-subsets, $3b_3$ appear only in three-times repeated blocks. Let $G = (V, E)$ be a graph called the *remainder* whose edges are the 2-subsets not appearing in three-times repeated blocks. The remainder has $3s$ edges when $v \equiv 1, 3 \pmod 6$, and has $3s + 1$ edges when $v \equiv 5 \pmod 6$. Moreover, the multigraph $3G$, obtained by repeating the edges of G each three times, must have a partition into triangles in which b_1 triangles appear once, b_2 triangles appear twice, and no triangle appears three times. First we observe that since $3G$ has a partition into triangles, all vertex degrees in $3G$ are even; hence all vertex degrees in G are even. Next we observe that no vertex degree in G is two, since that would require the inclusion of a three-times repeated block.

Lemma 2.1. *For* $v \equiv 1, 3 \pmod 6$, $\mathrm{SS}(v, 3) \subseteq P_v = \{m_v, m_v + 4, m_v + 6, m_v + 7, \cdots, 3m_v\}$.

Proof. Let $b_3 = m_v - s$. If $s \geqslant 6$, then $b^* \geqslant m_v + 6$ by (1) and (2) above. By Theorem B, we need only consider $s = 0$ and $s = 4$. When $s = 0$, $b_1 = b_2 = 0$ and $b^* = m_v$. When $s = 4$, the remainder must be a graph G with twelve edges and all vertex degrees even, and at least four. Then G cannot have as few as five or as many as seven vertices of nonzero degree; hence G is the unique 4-regular graph on six vertices. The graph $3G$ has only one partitioning up to isomorphism, and this partition has $b_1 = b_2 = 4$, giving $b^* = m_v + 4$. □

For $v \equiv 5 \pmod 6$, the situation is much more complicated, and candidate remainders were generated by computer. The observations above enable us to determine, for small values of s, all possible graphs which are candidates to be remainders, using a graph generation algorithm of Colbourn and Read [1].

Lemma 2.2. *For* $v \equiv 5 \pmod 6$, $\mathrm{SS}(v, 3) \subseteq Q_v = \{m_v + 7, m_v + 10, m_v + 11, \cdots, 3m_v + 1\}$.

Proof. Let $b_3 = m_v - s$. Since the remainder G has $3s + 1$ edges, we can strengthen (3) to $b_1 > b_2$. Hence for $s \geqslant 9$, we have $b^* \geqslant m_v + 10$. Now by Theorem B, we need only consider $s \in \{3, 6, 7, 8\}$. For $s = 3$, G must be a graph with ten edges which has vertices of even degree at least four. The only such graph is K_5, and the only partitioning of K_5 has $b_1 = 10$, $b_2 = 0$; this leads to $b^* = m_v + 7$. For $s \geqslant 6$, $b^* \geqslant m_v + 7$; hence it remains only to exclude $m_v + 8$ and $m_v + 9$. To do this, all graphs on 19, 22, and 25 edges having even vertex degrees at least four were generated; each was partitioned in all nonisomorphic ways into triangles, and each leads to a system with $b^* \geqslant m_v + 10$. □

It remains only to consider the necessary conditions for $v = 3$, 9. For $v = 3$, it is trivial that $\mathrm{SS}(v, 3) = \{1\}$. For $v = 9$, Khosrovshahi and Mahmoodian [11] show that $\mathrm{SS}(9, 3) \subseteq \{12, 18\text{–}36\}$. To show $19 \notin \mathrm{SS}(9, 3)$, we make some easy observations. Milici and Quattrochi [16] show that a TS(9, 3) can have b_3 three-times repeated blocks only for $b_3 \in \{0\text{–}4, 6, 12\}$. To obtain $b^* = 19$, we must have $b_3 = 6$, $b_2 = 5$, and $b_1 = 8$. The blocks repeated twice and three times comprise eleven disjoint blocks on nine elements; such a configuration is unique up to isomorphism, and leaves precisely the pairs of a triple $\{x, y, z\}$. Now twenty-four (not necessarily distinct) edges appear in blocks appearing just once. Nine of these edges are on the triple $\{x, y, z\}$, and at most ten of the remaining edges meet one of x, y, or z. But then the triple $\{x, y, z\}$ cannot be taken just once, and we have the required contradiction.

3. Recursive constructions

In this section, we develop techniques which enable us, given $\mathrm{SS}(v, 3)$, to determine information about $\mathrm{SS}(2v + 1, 3)$, $\mathrm{SS}(2v + 3, 3)$, and $\mathrm{SS}(2v + 7, 3)$. Our strategy is quite standard; these recursive methods enable us to settle the problem in subsequent sections by considering only "small" values of v. In the subsequent constructions, for an element x and multigraph G with edges $E(G)$, we use the notation $x \cdot G$ for the set of triples $\{\{x, a, b\} : \{a, b\} \in E(G)\}$.

Theorem 3.1. *Let* $v \equiv 1 \pmod 2$, $t \in \mathrm{SS}(v, 3)$, *and* $s \in \{v, v + 2, v + 3, \ldots, 3v\}$. *Then* $s(v + 1)/2 + t \in \mathrm{SS}(2v + 1, 3)$.

Proof. Let $(V, \mathscr{B})$ be a TS(v, 3) having t distinct blocks, and $V = \{1, \ldots, v\}$. Let X be a $(v + 1)$-set disjoint from V. Since $|X|$ is even, we can form a 1-factorization $\mathscr{F} = \{F_1, \ldots, F_v\}$ on X. Now define permutations σ and π so that the v multisets $\{\{i, \sigma(i), \pi(i)\} : 1 \leqslant i \leqslant v\}$ consist of r_1 multisets with all three elements the same, r_2 with two the same, and r_3 with all three different. It is an easy exercise to select σ and π so that $r_1 + 2r_2 + 3r_3 = s$ for any $v \leqslant s \leqslant 3v$ except for $s = v + 1$. Form $G_i = F_i \cup F_{\sigma(i)} \cup F_{\pi(i)}$ for $1 \leqslant i \leqslant v$. The $\{G_i\}$ form a 3-factorization of the complete multigraph $3K_{v+1}$ on X.

Let $\mathscr{C} = \bigcup_{i=1}^{v} i \cdot G_i$. Now form $(V \cup X, \mathscr{B} \cup \mathscr{C})$; this is a TS($2v + 1$, 3). By construction, $\mathscr{B}$ has t distinct triples; in addition, $\mathscr{C}$ has $s(v + 1)/2$ distinct triples. This completes the proof. □

Corollary 3.2. *If* $v \geq 7$ *and* $SS(v, 3) = P_v$ $(SS(v, 3) = Q_v)$, *then* $SS(2v + 1, 3) = P_{2v+1}$ $(SS(2v + 1, 3) = Q_{2v+1}$, *respectively*).

Theorem 3.1 will (eventually) handle all orders congruent to 3 (mod 4); we require somewhat more complicated constructions for the 1 (mod 4) case. First, we develop a $2v + 7$ construction, patterned on a similar construction due to Rosa [18].

Theorem 3.3. *Let* $k \in SS(v, 3)$, $s \in \{v, v + 2, v + 3, \ldots, 3v\}$, $\delta \in \{1, 2\}$ *and* $t \in SS(7, 3)$. *Then for* $v \geq 7$, $k + \frac{1}{2}s(v + 7) + \delta v + t \in SS(2v + 7, 3)$.

Proof. We form a $TS(2v + 7, 3)$ on element set $X \cup Y \cup Z$ where $|X| = |Y| = v$ and $|Z| = 7$. Let $X = \{x_1, \ldots, x_v\}$, $Y = \{y_1, \ldots, y_v\}$, and $Z = \{z_1, \ldots, z_7\}$. Let $(X, \mathcal{B}_1)$ be a $TS(v, 3)$ with k distinct blocks, and let $(Z, \mathcal{B}_2)$ be a $TS(7, 3)$ with t distinct blocks. Now write $m = \frac{1}{2}(v - 1)$ and form an $(m, 1)$-Langford sequence $\{(p_r, q_r): p_r - q_r = r,\ r = 1, 2, \ldots, m\}$. Let $S = \{y_i : i = p_r \text{ or } i = q_r \text{ and } r \geq 4\}$. Let $T = Y \backslash S$. Note that $|T| = 7$, and so we can write $T = \{y_{j_i} : 1 \leq i \leq 7\}$.

If $\delta = 1$, we take blocks $\mathcal{B}_3 = \{\{y_i, y_{i+1}, y_{i+3}\} : 0 \leq i \leq v$, each three times$\}$; if $\delta = 2$, we take each of these blocks only twice, together with the blocks $\{\{y_i, y_{i+2}, y_{i+3}\} : 0 \leq i < v\}$.

Now consider all pairs on $Y \cup Z$ not contained in blocks of $\mathcal{B}_3$. These form a v-regular graph which has a 1-factorization; for example, we can take the 1-factors

$$F_t = \{y_{p_r+t-1}, y_{q_r+t-1} : r = 4, \ldots, m\} \cup \{z_i, y_{j_i+t-1} : i = 1, \ldots, 7\}.$$

for $t = 1, \ldots, v$. Define permutations σ and π as in the proof of Theorem 3.1. Form $G_i = F_i \cup F_{\sigma(i)} \cup F_{\pi(i)}$. Finally, take $\mathcal{B}_4 = \bigcup_{i=1}^{v} x_i \cdot G_i$.

$(X \cup Y \cup Z, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4)$ is a $TS(2v + 7, 3)$ with the required number of distinct blocks. □

The technicality that $\delta \in \{1, 2\}$ in this proof limits its use in constructing designs with almost all blocks distinct. However, these cases are more easily handled using a $2v + 3$ construction.

Theorem 3.4. *Let* $v \geq 5$, $k \in SS(v, 3)$ *and* $s \in \{v + 6, v + 8, v + 9, \ldots, 3v + 2\}$. *Then* $\frac{1}{2}s(v + 3) + k \in SS(2v + 3, 3)$.

Proof. Let $(X, \mathcal{B}_1)$ be a $TS(v, 3)$ with k distinct blocks. Let $X = \{x_1, \ldots, x_v\}$ and $Y = \{y_1, \ldots, y_{v+3}\}$. Form $\mathcal{B}_2 = \{\{y_i, y_{i+1}, y_{i+2}\} : 1 \leq i \leq v + 3\}$. Let $\mathcal{G}$ be the graph whose edges are $\{\{y_i, y_j\} : i - j \geq 3 \pmod{v + 3}\}$. $\mathcal{G}$ has a 1-factorization $F_1, \ldots, F_{v-2}$ [6, 20]. Now let $s' = s - 8$, and select permutations σ and π so that $\{\{i, \sigma(i), \pi(i)\} : 1 \leq i \leq v - 2\}$ has r_1 sets with three equal elements, r_2 with two distinct, r_3 with three distinct, and $r_1 + 2r_2 + 3r_3 = s'$. Let $G_i = F_i \cup F_{\sigma(i)} \cup F_{\pi(i)}$ for $1 \leq i \leq v - 2$.

The edges unaccounted for on Y are as follows. Let $D_i = \{\{y_j, y_{j+i}\} : 1 \leq j \leq v+3\}$. The multigraph of edges which do not appear in $\mathscr{B}_2$ or in $\mathscr{G}$ is then $\mathscr{D} = D_1 \cup D_2 \cup D_2$. Now $\mathscr{D}$ is decomposable into two "prisms", which are 3-regular simple graphs. Let G_{v-1} and G_v be these two prisms.

Finally, form $\mathscr{B}_3 = \bigcup_{i=1}^{v} x_i \cdot G_i$. Now $(X \cup Y, \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3)$ is a TS$(2v+3, 3)$ with the required number of distinct blocks. □

There is some latitude in the proof of Theorem 3.4 which we have not exploited. The graph $\mathscr{D}$ will in general have more than one decomposition into 3-regular subgraphs, with differing number of repeated edges.

We also employ a general construction which enables us to fill a few "gaps" left by small exceptions. We define a *holey* triple system TS$(v:w, \lambda)$ to be a triple $(V, W, \mathscr{B})$ with $W \subseteq V$, $|W| = w$, $|V| = v$, and each pair of V which is not also a pair of W appearing in exactly λ blocks of $\mathscr{B}$, while pairs of W appear in no blocks. If a TS(w, λ) exists, filling the "hole" of the TS$(v:w, \lambda)$ with this design gives a TS(v, λ) with a subdesign; however, we do not require that this subdesign exist.

Combining the Doyen–Wilson theorem (for v, $w \equiv 1, 3 \pmod 6$) [3] with results of Mendelsohn and Rosa [15] establishes the following:

Lemma 3.5. *Suppose* $v \equiv 1, 3 \pmod 6$ *and* $w \equiv 1, 3 \pmod 6$, *or* $v \equiv w \equiv 5 \pmod 6$. *If* $v > 2w$, *a* TS$(v:w, 1)$ *exists.*

Corollary 3.6. *Suppose* $v \equiv 1, 3 \pmod 6$ *and* $w \equiv 1, 3 \pmod 6$, *or* $v \equiv w \equiv 5 \pmod 6$. *Suppose that* $v > 2w$. *If* $k \in$ SS$(w, 3)$, $m_v - m_w + k \in$ SS$(v, 3)$.

Proof. Form a TS$(v:w, 1)$ using Lemma 3.5. Triplicate each block of this holey triple system. Now on the hole W, place a copy of a TS$(w, 3)$ having k distinct blocks. □

4. Base cases for recursion

We first review the results previously known for small values.

Order v	SS$(v, 3)$	*Reference*
3	$\{1\}$	
5	$\{10\}$	
7	$\{7, 11, 13\text{–}15, 17\text{–}21\}$	[8]
9	$\{12, 18, 20\text{–}36\}$	[11], §2
11	$\{25, 28\text{–}55\}$	[13], §2

We now determine the spectrum for the next three values.

Lemma 4.1. $\mathrm{SS}(13, 3) = P_{13}$.

Proof. The Lindner–Rosa theorem [14] immediately gives $\{26, 30, 32\text{–}52\} \subseteq \mathrm{SS}(13, 3)$ by taking two TS(13, 1)'s with $52 - s$ blocks in common; a TS(13, 3) with s distinct blocks is obtained by taking two copies of one of the TS(13, 1)'s, and one copy of the other. That $\{53\text{–}78\} \subseteq \mathrm{SS}(13, 3)$ follows from computations available from the authors. In these computations, permutations of a TS(13, 1) are specified, which when taken together with the original system form TS(13, 3) designs with the required support sizes. □

Lemma 4.2. $\mathrm{SS}(15, 3) = P_{15}$.

Proof. For $s \in P_{15}$, $s \neq 44$, 52, Theorem 3.1 establishes that $s \in \mathrm{SS}(15, 3)$. For $s = 44$ or 52, we use Theorem 3.1 to construct a TS(15, 3) with $s - 3$ distinct blocks; we ensure that this system contains an SS(7, 3) with $(s - 3) - 28$ distinct blocks on elements $\{a, b, c, d, e, f, g\}$, and contains eight other elements $\{1, 2, 3, 4, 5, 6, 7, 8\}$. It is a simple matter to ensure further that the system contains the four blocks $\{a, b, c\}$, $\{a, 1, 2\}$, $\{b, 1, 3\}$, $\{c, 2, 3\}$ so that the first of these appears only once, while the other three are three-times repeated blocks. Replacing once copy of each of these four blocks by the four blocks $\{a, b, 1\}$, $\{a, c, 2\}$, $\{b, c, 3\}$, $\{1, 2, 3\}$ then gives a TS(15, 3) with support size s. □

The last small case needed is $v = 17$. We cannot apply Theorem 3.3, but can apply Theorem 3.4 to give

Lemma 4.3. $\{72, 76, 78\text{–}80, 82\text{–}86, 88\text{–}90, 92\text{–}136\} \subseteq \mathrm{SS}(17, 3)$.

We can improve on this by observing that the graph $\mathscr{D}$ in the proof of Theorem 3.4 can be decomposed in a number of ways.

Lemma 4.4. $\{66, 68, 70, 73, 74, 75, 77, 81, 87, 91\} \subseteq \mathrm{SS}(17, 3)$.

Proof. We modify the proof of Theorem 3.4 to obtain a 3-factorization of $\mathscr{D}$ having c repeated edges for $c = 2, 4, 6$. We take $Y = \{0, 1, \ldots, 9\}$. Then $\mathscr{D}$ contains edges of the form $\{i, i + 1\}$ each once, and edges of the form $\{i, i + 2\}$ each twice. The three required 3-factorizations of $\mathscr{D}$ are given here. We give the factor G_{v-1}; the factor G_v is $\mathscr{D} \setminus G_{v-1}$.

c	G_{v-1}														
2	02	02	08	13	13	19	24	35	45	46	57	67	68	79	89
4	02	02	08	13	13	19	24	35	46	46	57	57	68	79	89
6	02	02	08	13	13	19	23	45	46	46	57	57	68	79	89

These three 3-factorizations can be substituted in the proof to reduce the number of distinct triples in Theorem 3.4 (for $v = 17$) by 2, 4, or 6 blocks, respectively. □

It is important to remark that Corollary 3.6 demonstrates that $52 \in \mathrm{SS}(17, 3)$. We give an explicit construction of a TS(17, 3) $\mathcal{R}$ having 52 distinct blocks here. $\mathcal{R}$ has 17 points, $\{a, b, c, d, e, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B\}$. On $\{a, b, c, d, e\}$, $\mathcal{R}$ has the ten blocks of the TS(5, 3). Its remaining blocks are shown here; each is repeated three times in $\mathcal{R}$:

012	036	*a*04	*b*07	*c*0*B*	*d*09	*e*08
05*A*	138	*a*17	*b*14	*c*1*A*	*d*1*B*	*e*16
159	279	*a*2*A*	*b*23	*c*26	*d*24	*e*25
28*B*	345	*a*3*B*	*b*5*B*	*c*39	*d*37	*e*3*A*
46*B*	47*A*	*a*5*B*	*b*6*A*	*c*48	*d*56	*e*49
678	9*AB*	*a*69	*b*89	*c*57	*d*8*A*	*e*7*B*

Many small changes can be made to the structure of $\mathcal{R}$ by removing small subsets of blocks and replacing them with a mutually balanced set (making a "trade"). We list some small trades for $\mathcal{R}$ here:

Trade	*Blocks removed*				*Blocks added*			
S_1	*a*04	*b*07	*a*17	*b*14	*a*07	*b*04	*a*14	*b*17
S_2	678	9*AB*	279		68*B*	79*A*	28*B*	
	28*B*	47*A*	46*B*		278	4*AB*	467	
S_3	*b*07	*c*0*B*	*b*5*B*	*c*57	*b*0*B*	*c*07	*b*57	*c*5*B*
S_4	*a*17	*d*1*B*	*a*3*B*	*d*37	*a*1*B*	*d*17	*a*37	*d*3*B*
S_5	*a*2*A*	*b*23	*a*3*B*	*b*5*B*	*a*23	*b*2*A*	*a*5*B*	*b*3*B*
	*a*58	*b*6*A*	*a*69	*b*89	*a*6*A*	*b*58	*a*89	*b*69
S_6	*abc*	6*bA*	*a*2*A*	6*c*2	*abA*	*bc*6	*ac*2	26*A*
S_7	*acd*	*d*09	*a*69		0*cd*	*ad*9	069	
	036	*a*3*B*	*c*0*B*		*a*36	03*B*	*acB*	

With these trades, we can prove

Lemma 4.5. $\mathrm{SS}(17, 3) = Q_{17}$.

Proof. We handle all cases not yet settled by Lemmas 4.3 and 4.4 here. $\mathcal{R}$ has 52 distinct blocks. We show how to carry out a sequence of trades for the remaining values shown. Each is a sequence of trades to be applied to $\mathscr{R}$:

Support Size	*Trades Used*
55	S_6
56	S_1
57	S_7
58	S_2
59	S_1, S_1, S_3
60	S_1, S_3
61	S_6, S_2
62	S_1, S_2
63	S_1, S_3, S_4
64	S_5, S_1
65	S_1, S_1, S_3, S_2
67	S_5, S_5, S_3
69	S_1, S_3, S_4, S_2

To obtain 71 distinct blocks, interchange d and e in one copy of each block containing d and e of $\mathscr{R}$; then apply trades S_1 twice followed by S_3. □

Now we are prepared to complete the recursion.

5. Proof of the main theorem

Necessity is established in Section 2. Here we prove the sufficiency of the conditions. The constructions in Section 4 settle all cases for $v \leqslant 17$. So let $v \geqslant 19$, and assume that the theorem has been established for all $17 \leqslant x < v$.

If $v \equiv 1, 9 \pmod{12}$, write $v = 2x + 7$. Now $x \geqslant 7$ and $x \equiv 1, 3 \pmod 6$. Theorems 3.3 and 3.4 show that $\mathrm{SS}(v, 3) = P_v$, except in the case $x = 7, 9$. For $v = 25$, these theorems do not hand the values $m_{25} + 4$ and $m_{25} + 7$; these are handled by applying Corollary 3.6 with $v = 25$ and $w = 7$. Similarly, for $v = 21$, the theorems do not handle $m_v + 9$; applying Corollary 3.6 with $v = 21$ and $w = 9$ handles this case.

If $v \equiv 3, 7 \pmod{12}$, write $v = 2x + 1$. Now $x \geqslant 9$ and $x \equiv 1, 3 \pmod 6$. Corollary 3.2 handles this case, with two exceptions when $v = 19$. These are handled by applying Corollary 3.6 with $v = 19$ and $w = 7$.

If $v \equiv 5 \pmod{12}$, write $v = 2x + 7$. Now $x \equiv 5 \pmod 6$ and $x \geqslant 11$. Theorems 3.3 and 3.4 establish that $\mathrm{SS}(v, 3) = Q_v$.

If $v \equiv 11 \pmod{12}$, write $v = 2x + 1$. Now $x \equiv 5 \pmod 6$ and $x \geqslant 11$. Hence Corollary 3.2 establishes $\mathrm{SS}(v, 3) = Q_v$. □

6. Concluding remarks

A complete determination of SS(v, λ) is now done for $\lambda \leq 3$; extending this to all λ would be a worthwhile goal. The constructions of [19] and those here all generalize in a natural way; in one sense, the problem becomes easier because the spectra for small values of v contain long intervals which aid in the recursion. One serious obstacle remains, however. While a $2v + 1$ construction is known which does not introduce repeated blocks [2], this approach would require $2v + a$ constructions for small values of a which also do not introduce repeated blocks. The complete determination of SS(v, λ) therefore remains a problem of much interest.

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